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THE DETERMINATION OF THE COEFFICIENTS IN THE LEGENDRE POLYNOMIAL EXPANSION OF THE GRAVITATIONAL POTENTIAL OF THE EARTH

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ABSTRACT: Using the solution of Stokes' problem the authors find exact expressions for the coefficients in the Legendre polynomial expansion of the potential of the adjusted gravity field of the earth when a Clairaut ellipsoid is taken as the level surface of the gravity field.

THE DETERMINATION OF THE COEFFICIENTS IN THE LEGENDRE POLYNOMIAL EXPANSION OF THE GRAVITATIONAL POTENTIAL OF THE EARTH

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Based on the solution of Stokes' problem [1-4], exact expressions are found for the coefficients in the Legendre polynomial expansion of the potential of the adjusted gravity field of the earth when a Clairaut ellipsoid is taken as the level surface of the gravity field. /749*

1. The solution of Stokes' problem, with a Clairaut ellipsoid [4] as the level surface of the earth's gravity field, reduces to the following expression for the gravitational field potential V [1-3] in the right-hand orthogonal coordinate system Oxyz tied to the earth (the origin O of this coordinate system coincides with the center of the earth, the z-axis is directed along the axis of rotation of the earth):

$$V(x, y, z) = -AP(x^2 + y^2) - BQz^2 + CR \tag{1.1}$$

Here

$$P = \operatorname{arc\,tg} \varepsilon' - \frac{\varepsilon'}{1 + \varepsilon'^2}, \quad Q = \varepsilon' - \operatorname{arc\,tg} \varepsilon', \quad R = \operatorname{arc\,tg} \varepsilon' \tag{1.2}$$

where ε' is the second eccentricity of an ellipsoid that is confocal with the Clairaut ellipsoid and passes through the point at which the potential is defined; A, B and C are constants.

The quantity ε' is defined from the equality

$$\varepsilon' = [(a^2 - b^2) / (b^2 + \nu)]^{1/2} \tag{1.3}$$

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*Numbers in the margin indicate pagination in the foreign text.

where a and b are the semi-major and semi-minor axes of the Clairaut ellipse and ν is the positive root of the equation

$$\frac{x^2+y^2}{a^2+\nu}+\frac{z^2}{b^2+\nu}=1 \tag{1.4}$$

The constants A, B and C in formula (1.1) are found from the relationships

$$A=\frac{u^2(1+\varepsilon^2)}{2[(3+\varepsilon^2)\operatorname{arc\,tg}\varepsilon-3\varepsilon]},\qquad B=2A$$
$$C=\frac{a^2}{\varepsilon}\left\{\frac{g_e+u^2a}{a}+\frac{u^2(1+\varepsilon^2)(\varepsilon-\operatorname{arc\,tg}\varepsilon)}{(3+\varepsilon^2)\operatorname{arc\,tg}\varepsilon-3\varepsilon}\right\} \tag{1.5}$$

Here $\varepsilon = (a^2 - b^2)^{1/2}/b$ is the second eccentricity of the Clairaut ellopsoid, u is the angular rate of rotation of the earth, g_e is the acceleration of gravity at the equator. /750

Relationships (1.1)-(1.5) yield an implicit expression for the potential V(x, y, z) which is not convenient for practical computations, and therefore the gravity field potential is usually [2-5] represented in the form of a Legendre polynomial expansion

$$V(r,\varphi)=\sum_{n=0}^{\infty}A_n\left(\frac{a}{r}\right)^{n+1}P_n(\sin\varphi) \tag{1.6}$$

where the distance r from the center of the earth and the latitude φ are the geocentric coordinates of the point for which the potential is computed, $P_n(\sin\varphi)$ is the n-th order Legendre polynomial and A_n are constant coefficients in the expansion.

The exact solution (1.1)-(1.5) of Stokes' problem is not usually used when determining the coefficients A_n of the expansion (1.6). They are determined directly, due to the circumstance that the surface of the Clairaut ellipse is the level surface of the gravity field. In addition, approximate expressions for the first coefficients A_n are obtained as series expansions in the small parameters

$$\varepsilon^2=\frac{a^2-b^2}{b^2}\ \left(\text{or } e^2=\frac{a^2-b^2}{a^2}\right)\ \text{ and }\ q=\frac{u^2a}{g_e} \tag{1.7}$$

The first (and perhaps the second) terms of the expansion are obtained relatively easily [3, 4]. It also appears possible in this way to set up recursion relationships for finding the series expansions for the coefficients A_n of any number sequentially, to any desired accuracy. However, the method for establishing these relationships and the relationships themselves are very complicated since the problem, all things considered, leads to the solution of an infinite (triangular) system of linear algebraic equations.

The determination of the first coefficients A_n of the potential expansion (1.6) on the basis of an exact solution of Stokes' problem is given in [6]. But in this paper they are also defined as expansions in powers of small parameters (1.7).

2. Converting to spherical coordinates (geocentric) r, φ, λ in formula (1.1) according to the relations $x = r\cos\varphi\cos\lambda$, $y = r\cos\varphi\sin\lambda$, $z = r\sin\varphi$, we write the equation in the following form:

$$V = -A\left(\frac{r}{a}\right)^2 (P\cos^2\varphi + 2Q\sin^2\varphi) + CR \tag{2.1}$$

From (1.3) and (1.4), introducing the notation

$$t = \frac{a}{r}, \qquad e = \frac{\varepsilon}{\sqrt{1+\varepsilon^2}} \tag{2.2}$$

(e is the first eccentricity of the Clairaut ellopsoid), we obtain the following equation defining e':

$$\varepsilon'^4 \sin^2\varphi - \varepsilon'^2 (e^2t^2 - 1) - e^2t^2 = 0 \tag{2.3}$$

from which

$$\varepsilon' = \frac{1}{\sqrt{2}\sin\varphi}\left[e^2t^2 - 1 + \sqrt{(e^2t^2-1)^2 + 4e^2t^2\sin^2\varphi}\right]^{1/2} \tag{2.4}$$

From the last equality of (1.2) and Eq. (2.4) we find

$$\frac{\partial R}{\partial t} = \frac{e}{\sqrt{2}}\left[\frac{\sqrt{(e^2t^2-1)^2 + 4e^2t^2\sin^2\varphi} - e^2t^2 + 1}{(e^2t^2-1)^2 + 4e^2t^2\sin^2\varphi}\right]^{1/2} \tag{2.5}$$

On going over to complex numbers, expression (2.5) for $\partial R/\partial t$ can be represented in the form

$$\frac{\partial R}{\partial t} = \frac{e}{2}\left[\frac{1}{\sqrt{1 - 2\sin\varphi\,(iet) + (iet)^2}} + \frac{1}{\sqrt{1 - 2\sin\varphi\,(-iet) + (-iet)^2}}\right], \qquad i = \sqrt{-1} \tag{2.6}$$

Since the function $(1 - 2x\tau + \tau^2)^{1/2}$ is the generating function for the Legendre /751
polynomials $P_n(x)$, i.e.

$$\frac{1}{\sqrt{1 - 2x\tau + \tau^2}} = \sum_{n=0}^{\infty} \tau^n P_n(x) \tag{2.7}$$

it follows from (2.6) that

$$\frac{\partial R}{\partial t} = \frac{e}{2}\sum_{n=0}^{\infty} P_n(\sin\varphi)\,[(iet)^n + (-1)^n (iet)^n] = e\sum_{k=0}^{\infty}(-1)^k (et)^{2k} P_{2k}(\sin\varphi) \tag{2.8}$$

and, consequently, $(t = a/r)$:

$$R = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \left(e \frac{a}{r}\right)^{2k+1} P_{2k}(\sin\varphi) + C_1(\varphi) \tag{2.9}$$

where $C_1(\varphi)$ is some function of latitude.

Further, from Eqs. (1.2) and (2.4), taking (2.2) into account, we find

$$\frac{\partial}{\partial t}(P\cos^2\varphi + 2Q\sin^2\varphi) = 2e^2t^2\frac{\partial R}{\partial t} \tag{2.10}$$

From this and (2.8), again converting from t to a/r, we find

$$\left(\frac{r}{a}\right)^2 (P\cos^2\varphi + 2Q\sin^2\varphi) = 2e^2 \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+3}\left(e\frac{a}{r}\right)^{2k+1} P_{2k}(\sin\varphi) + \left(\frac{r}{a}\right)^2 C_2(\varphi) \tag{2.11}$$

where $C_2(\varphi)$ is also some function of latitude.

3. On substituting expressions (2.9) and (2.11) into Eq. (2.1) we arrive at the following expression for the potential V:

$$V = \sum_{k=0}^{\infty}\left[-\frac{2Ae^2}{2k+3} + \frac{C}{2k+1}\right](-1)^k\left(e\frac{a}{r}\right)^{2k+1} P_{2k}(\sin\varphi) + CC_1(\varphi) - A\left(\frac{r}{a}\right)^2 C_2(\varphi) \tag{3.1}$$

Since the potential V satisfies the condition

$$\lim rV = \text{const}, \quad r \to \infty \tag{3.2}$$

then in (3.1), $C_1(\varphi) \equiv 0$, $C_2(\varphi) \equiv 0$ and consequently we obtain the following expression for the earth's gravitational potential:

$$V(r, \varphi) = \sum_{k=0}^{\infty}\left[-\frac{2Ae^2}{2k+3} + \frac{C}{2k+1}\right](-1)^k\left(e\frac{a}{r}\right)^{2k+1} P_{2k}(\sin\varphi) \tag{3.3}$$

From expression (1.5) and the last equality of (1.7) we obtain the following values for the constants A and C:

$$A = \frac{g_e q a(1+\varepsilon^2)}{2[(3+\varepsilon^2)\,\mathrm{arc\,tg}\,\varepsilon - 3\varepsilon]}$$

$$C = \frac{g_e a}{\varepsilon}\left\{1 + q\left[1 + \frac{(1+\varepsilon^2)(\varepsilon - \mathrm{arc\,tg}\,\varepsilon)}{(3+\varepsilon^2)\,\mathrm{arc\,tg}\,\varepsilon - 3\varepsilon}\right]\right\} \tag{3.4}$$

In this way

$$V = \sum_{k=0}^{\infty} A_{2k} \left(\frac{a}{r}\right)^{2k+1} P_{2k} (\sin \varphi) \tag{3.5}$$

where /752

$$A_{2k} = (-1)^k e^{2k+1} \left(-\frac{2Ae^2}{2k+3} + \frac{C}{2k+1}\right) \tag{3.6}$$

and the constants A and C are defined by Eqs. (3.4).

After substituting the values of A and C into (3.6) and making use of the second equality of (2.2), we find, finally:

$$A_{2k} = \frac{g_e a \varepsilon^{2k}}{(1+\varepsilon^2)^{1/2\,(2k+1)}} \frac{(-1)^k}{2k+1} \left\{1 + 2q \frac{(2k+1)(\operatorname{arc\,tg} \varepsilon - \varepsilon) + \varepsilon^3}{(2k+3)\left[(3+\varepsilon^2)\operatorname{arc\,tg} \varepsilon - 3\varepsilon\right]}\right\} \tag{3.7}$$

Together with (3.5) an expression for the potential can be written in the form:

$$V = J_0 \left[\frac{a}{r} + \sum_{k=1}^{\infty} J_{2k} \left(\frac{a}{r}\right)^{2k+1} P_{2k} (\sin \varphi)\right] \tag{3.8}$$

Then

$$\begin{gathered} J_0 = \frac{g_e a}{(1+\varepsilon^2)^{1/2}} \left\{1 + \frac{2q}{3} \frac{3(\operatorname{arc\,tg} \varepsilon - \varepsilon) + \varepsilon^3}{(3+\varepsilon^2)\operatorname{arc\,tg} \varepsilon - 3\varepsilon}\right\} \\ J_{2k} = \frac{(-1)^k \varepsilon^{2k}}{(2k+1)(1+\varepsilon^2)^k} \times \\ \times \left\{1 - \frac{4k\varepsilon^3 q}{(2k+3)\left[3(3+\varepsilon^2+2q)\operatorname{arctg} \varepsilon - 9\varepsilon - 6\varepsilon q + 2\varepsilon^3 q\right]}\right\} \end{gathered} \tag{3.9}$$

It is easily verfied that approximate expressions for the coefficients J_{2k} can also be established by expanding formulas (3.9) in powers of ε^2 and q:

$$\begin{gathered} J_0 = g_e \left(1 - {}^1/_2\varepsilon^2 + {}^3/_2 q + {}^3/_8 \varepsilon^4 - {}^{15}/_{28}\varepsilon^6 + {}^{943}/_{2352}\varepsilon^4 q \ldots\right) \\ J_2 = -{}^1/_3\varepsilon^2 + {}^1/_3 q + {}^1/_3\varepsilon^4 - {}^1/_{21}\varepsilon^3 q - {}^1/_2 q^2 - {}^1/_3\varepsilon^6 + {}^3/_4 q^3 + {}^8/_{147}\varepsilon^4 q \ldots \\ J_4 = \varepsilon^2\left({}^1/_5\varepsilon^2 - {}^2/_7 q - {}^2/_5\varepsilon^4 + {}^3/_7 q^2 + {}^{16}/_{49}\varepsilon^2 q \ldots\right) \\ J_6 = \varepsilon^4 \left({}^5/_{21} q - {}^1/_7 \varepsilon^2 \ldots\right) \end{gathered} \tag{3.10}$$

The first terms of the expansions obtained for J_0, J_2 and J_4 agree with those given in [4, 6].

If we take [1, 4, 7, 8] u = 7.29212 x 10^{-5} 1/sec and g_e = 978.049 cm/sec^2, the computed values of these coefficients, for the parameters of the Krasovski ellipsoid (a = 6738245 m, e^2 = 0.006693422) become:

$$J_0/a = 979.846\ \text{м}.\ J_2 = -1082.24\cdot10^{-6}.\ J_4 = 2.4\cdot10^{-6}.\ J_6 = -6.3\cdot10^{-9}:$$

and, for the parameters of a Clark ellipsoid (a = 6378206 m, e^2 = 0.00676817), respectively

$$J_0/a = 979.809\ \text{м}.\ J_2 = -1107.19\cdot10^{-6}.\ J_4 = 2.5\cdot10^{-6}.\ J_6 = -6.3\cdot10^{-9}.$$

REFERENCES

1. Mikhaylov, A. A. Kurs gravimetrii i teorii figury Zemli (Course in Gravimetry and the Theory of the Figure of the Earth.) Ed. Bureau GUGK with SNK SSR, Moscow, Gostekhizdat, 1939.
2. Duboshin, N. G. Teoriya prityazheniya (Theory of Gravitational Attraction.) Moscow, Fizmatgiz, 1961.
3. Idel'son, N. I. Teoriya potensiala i yeye prilozheniye k voprosam geofiziki (Potential Theory and its Application to Geophysical Problems.) Leningrad-Moscow, Gostekhizdat, 1932.
4. Grushinskiy, N. P. Teoriya figury Zemli (Theory of the Figure of the Earth.) Moscow, Fizmatgiz, 1963.
5. Demin, V. G. Dvizheniye iskusstvennogo sputnika Zemli v netsentral'nom pole tygoteniya (The Motion of Artificial Earth Satellites in the Central Gravity Field.) Moscow, "Nauka", 1968.
6. Andreyev, V. D. K resheniyu problemy Stoksa dlya urovennoy poverkhnosti, zadannoy v forme sferoida (The Solution of Stokes' Problem for a Level Surface in the Form of a Spheriod.) PMM, Vol. 30, No. 2, 1966.
7. Graur, A. V. Matematicheskaya kartografiya (Mathematical Cartography.) Pub; Leningrad Univ., 1956.
8. Madelung, E. (Matematicheskiy apparat fiziki.) Moscow, Fizmatgiz, 1960.

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